Problem 16.9

The motion of a finite string, fixed at both ends, was determined by the wave equation (16.19) and the boundary conditions (16.20). We solved these by looking for a solution that was sinusoidal in time. A different, and rather more general, approach to problems of this kind is called **separation of variables**. In this approach, we seek solutions of (16.19) with the *separated* form $u(x,t) = X(x)T(t)$, that is, solutions that are a simple product of one function of x and a second of t. [As usual, there's nothing to stop us trying to find a solution of this form. In fact, there is a large class of problems (including this one) where this approach is known to produce solutions, and enough solutions to allow expansion of *any* solution.] **(a)** Substitute this form into (16.19) and show that you can rewrite the equation in the form $T''(t)/T(t) = c^2 X''(x)/X(x)$. **(b)** Argue that this last equation requires that both sides of this equation are separately equal to the same constant (call it K). It can be shown that K has to be negative.[26] Use this to show that the function $T(t)$ has to be sinusoidal—which establishes (16.21) and we're back to the solution of Section 16.3. The method of separation of variables plays an important role in several areas, notably quantum mechanics and electromagnetism.

Solution

The aim here is to solve the wave equation on a finite interval with fixed ends and a specified shape and speed initially.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \ -\infty < t < \infty$$

$$u(0,t) = 0$$

$$u(L,t) = 0$$

$$u(x,0) = f(x)$$

$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

Because the PDE and boundary conditions are linear and homogeneous, the method of separation of variables can be applied here: Assume a product solution of the form $u(x,t) = X(x)T(t)$ and plug it into the PDE

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \rightarrow \quad \frac{\partial^2}{\partial t^2}[X(x)T(t)] = c^2 \frac{\partial^2}{\partial x^2}[X(x)T(t)] \quad \rightarrow \quad X(x)T''(t) = c^2 X''(x)T(t)$$

and the boundary conditions.

$$u(0,t) = 0 \quad \rightarrow \quad X(0)T(t) = 0 \quad \rightarrow \quad X(0) = 0$$

$$u(L,t) = 0 \quad \rightarrow \quad X(L)T(t) = 0 \quad \rightarrow \quad X(L) = 0$$

Separate variables in the PDE by dividing both sides by $c^2 X(x)T(t)$.

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)}$$

[26] Actually this isn't hard to show. Look at the equation $X''(x)/X(x) = K/c^2$. You can show that if $K > 0$ there are no solutions satisfying the boundary conditions that $X(0) = X(L) = 0$.

c^2 is only a constant and can be put on either side; the final answer is the same regardless, but it is customary to group constants with t whenever possible. The only way a function of t can be equal to a function of x is if both sides are equal to a constant K.

$$\frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} = K$$

As a result of using the method of separation of variables, the wave equation has reduced to two ODEs, one in x and one in t.

$$\left.\begin{aligned} \frac{T''(t)}{c^2T(t)} &= K \\ \frac{X''(x)}{X(x)} &= K \end{aligned}\right\}$$

Values of K for which the boundary conditions are satisfied are called the eigenvalues, and the nontrivial solutions associated with them are called the eigenfunctions.

$$X''(x) = KX(x), \quad X(0) = 0, \ X(L) = 0$$

Check to see if there are positive eigenvalues: $K = \mu^2$.

$$X''(x) = \mu^2 X(x)$$

The general solution for $X(x)$ can be written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \mu x + C_2 \sinh \mu x$$

Apply the boundary conditions to determine C_1 and C_2.

$$X(0) = C_1 = 0$$

$$X(L) = C_1 \cosh \mu L + C_2 \sinh \mu L = 0$$

Since $C_1 = 0$, the second equation becomes $C_2 \sinh \mu L = 0$. There's no nonzero value of μ for which this equation is satisfied, so it's necessary that $C_2 = 0$ as well. This results in the trivial solution $X(x) = 0$; consequently, there are no positive eigenvalues. Check to see if zero is an eigenvalue: $K = 0$.

$$X''(x) = 0$$

The general solution for $X(x)$ is obtained by integrating with respect to x twice.

$$X(x) = C_3 x + C_4$$

Apply the boundary conditions to determine C_3 and C_4.

$$X(0) = C_4 = 0$$

$$X(L) = C_3 L + C_4 = 0$$

Since $C_4 = 0$, the second equation becomes $C_3 L = 0$. Solving this for C_3 gives $C_3 = 0$. As a result, the trivial solution $X(x) = 0$ is obtained, which means zero is not an eigenvalue. Check to see if there are negative eigenvalues: $K = -\gamma^2$.

$$X''(x) = -\gamma^2 X(x)$$

The general solution for $X(x)$ can be written in terms of sine and cosine.

$$X(x) = C_5 \cos \gamma x + C_6 \sin \gamma x$$

Apply the boundary conditions to determine C_5 and C_6.

$$X(0) = C_5 = 0$$

$$X(L) = C_5 \cos \gamma L + C_6 \sin \gamma L = 0$$

Since $C_5 = 0$, the second equation becomes $C_2 \sin \gamma L = 0$. In this case, there are nonzero values of γ that satisfy the equation.

$$\sin \gamma L = 0$$

$$\gamma L = n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$\gamma = \frac{n\pi}{L}$$

There are negative eigenvalues $K = -\gamma^2 = -n^2\pi^2/L^2$ for $n = 1, 2, \ldots$, and the eigenfunctions associated with them are

$$\begin{aligned} X(x) &= C_5 \cos \gamma x + C_6 \sin \gamma x \\ &= C_6 \sin \frac{n\pi x}{L}, \end{aligned}$$

where C_6 remains arbitrary. Note that n only takes positive values because negative values lead to redundant eigenvalues, and $n = 0$ leads to the zero eigenvalue. Solve the ODE for t now with these values of K.

$$\frac{T''(t)}{c^2 T(t)} = -\frac{n^2\pi^2}{L^2}$$

Multiply both sides by $c^2 T(t)$.

$$T''(t) = -\frac{n^2\pi^2 c^2}{L^2} T(t)$$

The general solution for $T(t)$ is written in terms of sine and cosine.

$$T(t) = C_7 \cos \frac{n\pi ct}{L} + C_8 \sin \frac{n\pi ct}{L}$$

According to the principle of superposition, the general solution for $u(x,t)$ is a linear combination of the product solutions $X_n(x)T_n(t)$ over all the eigenvalues.

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x) T_n(t)$$

Therefore,

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L} \right) \sin \frac{n\pi x}{L}.}$$

Differentiate it with respect to t.

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \left(-A_n \frac{n\pi c}{L} \sin \frac{n\pi ct}{L} + B_n \frac{n\pi c}{L} \cos \frac{n\pi ct}{L} \right) \sin \frac{n\pi x}{L}$$

In order to determine A_n, set $t = 0$ in the general solution.

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = f(x)$$

Multiply both sides by $\sin(m\pi x/L)$, where m is another positive number.

$$\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L} = f(x)\sin\frac{m\pi x}{L}$$

Integrate both sides over the interval of x where the string is (from 0 to L).

$$\int_0^L \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \int_0^L f(x)\sin\frac{m\pi x}{L}\,dx$$

Move the constants in front of the integral.

$$\sum_{n=1}^{\infty} A_n \int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \int_0^L f(x)\sin\frac{m\pi x}{L}\,dx$$

Use the product-to-sum formulas for sine-sine.

$$\sum_{n=1}^{\infty} A_n \int_0^L \frac{1}{2}\left[\cos\left(\frac{n\pi x}{L} - \frac{m\pi x}{L}\right) - \cos\left(\frac{n\pi x}{L} + \frac{m\pi x}{L}\right)\right] dx = \int_0^L f(x)\sin\frac{m\pi x}{L}\,dx$$

Factor the cosine arguments.

$$\sum_{n=1}^{\infty} A_n \int_0^L \frac{1}{2}\left[\cos\frac{(n-m)\pi x}{L} - \cos\frac{(n+m)\pi x}{L}\right] dx = \int_0^L f(x)\sin\frac{m\pi x}{L}\,dx$$

If $n \neq m$, the integral on the left evaluates to zero.

$$\frac{1}{2}\left[\frac{L}{(n-m)\pi}\sin\frac{(n-m)\pi x}{L} - \frac{L}{(n+m)\pi}\sin\frac{(n+m)\pi x}{L}\right]\Bigg|_0^L = 0$$

Consequently, there's only one term in the infinite series that isn't zero—the one in which $n = m$.

$$A_n \int_0^L \frac{1}{2}\left[\cos\frac{(n-n)\pi x}{L} - \cos\frac{(n+n)\pi x}{L}\right] dx = \int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

Simplify the left side.

$$\frac{A_n}{2}\int_0^L \left(1 - \cos\frac{2n\pi x}{L}\right) dx = \int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$\frac{A_n}{2}\left(x - \frac{L}{2n\pi}\sin\frac{2n\pi x}{L}\right)\Bigg|_0^L = \int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$\frac{A_n}{2}(L) = \int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

Therefore,

$$\boxed{A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.}$$

In order to determine B_n, set $t = 0$ in the formula for $\partial u/\partial t$.

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L}\sin\frac{n\pi x}{L} = g(x)$$

Multiply both sides by $\sin(p\pi x/L)$, where p is another positive number.

$$\sum_{n=1}^{\infty} B_n \frac{n\pi c}{L}\sin\frac{n\pi x}{L}\sin\frac{p\pi x}{L} = g(x)\sin\frac{p\pi x}{L}$$

Integrate both sides over the interval of x where the string is (from 0 to L).

$$\int_0^L \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L}\sin\frac{n\pi x}{L}\sin\frac{p\pi x}{L}\,dx = \int_0^L g(x)\sin\frac{p\pi x}{L}\,dx$$

Move the constants in front of the integral.

$$\sum_{n=1}^{\infty} B_n \frac{n\pi c}{L}\int_0^L \sin\frac{n\pi x}{L}\sin\frac{p\pi x}{L}\,dx = \int_0^L g(x)\sin\frac{p\pi x}{L}\,dx$$

Use the product-to-sum formulas for sine-sine.

$$\sum_{n=1}^{\infty} B_n \frac{n\pi c}{L}\int_0^L \frac{1}{2}\left[\cos\left(\frac{n\pi x}{L} - \frac{p\pi x}{L}\right) - \cos\left(\frac{n\pi x}{L} + \frac{p\pi x}{L}\right)\right]dx = \int_0^L g(x)\sin\frac{p\pi x}{L}\,dx$$

Factor the cosine arguments.

$$\sum_{n=1}^{\infty} B_n \frac{n\pi c}{L}\int_0^L \frac{1}{2}\left[\cos\frac{(n-p)\pi x}{L} - \cos\frac{(n+p)\pi x}{L}\right]dx = \int_0^L g(x)\sin\frac{p\pi x}{L}\,dx$$

If $n \neq p$, the integral on the left evaluates to zero.

$$\frac{1}{2}\left[\frac{L}{(n-p)\pi}\sin\frac{(n-p)\pi x}{L} - \frac{L}{(n+p)\pi}\sin\frac{(n+p)\pi x}{L}\right]\Bigg|_0^L = 0$$

Consequently, there's only one term in the infinite series that isn't zero—the one in which $n = p$.

$$B_n \frac{n\pi c}{L}\int_0^L \frac{1}{2}\left[\cos\frac{(n-n)\pi x}{L} - \cos\frac{(n+n)\pi x}{L}\right]dx = \int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

This integral on the left side is the same as before.

$$B_n \frac{n\pi c}{L}\frac{1}{2}(L) = \int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

Therefore,

$$\boxed{B_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx.}$$